PROOF OF THE VOLUME CONJECTURE FOR TORUS KNOTS

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ABSTRACT. The volume conjecture, formulated recently by H. Murakami and J. Murakami, is proved for the case of torus knots.

1. INTRODUCTION

In the recent paper [6] H. Murakami and J. Murakami showed that the "quantum dilogarithm" knot invariant, introduced in [1, 2], is a special case of the colored Jones invariant (polynomial) associated with the quantum group $SU(2)_q$. Using the connection of the quantum dilogarithm invariant with the hyperbolic volume of knot's complement, conjectured in [3], they also proposed the "volume conjecture": for any knot the above mentioned specification of the colored Jones invariant in a certain limit gives the simplicial volume (or Gromov norm) of the knot. It is remarkable that this conjecture implies that a knot is trivial if and only if its colored Jones invariants are trivial, see [6].

The purpose of this paper is to prove the volume conjecture for the case of torus knots. To formulate our result, let us first recall the form of the colored Jones invariant for torus knots [5, 7].

The colored Jones invariant $J_{K,k}(h)$ of a framed knot K is a Laurent polynomial in $q = e^h$ depending on the 'color' k, the dimension of a $SU(2)_q$-module. Let m, p be mutually prime positive integers. Denote $L \equiv O_{m,p}$ the (m, p) torus knot obtained as the (m, p) cable about the unknot with zero framing (see [5] for the precise definition). Then the colored Jones invariant of L has the following explicit form:

$$(1) \qquad 2\sinh(kh/2)\frac{J_{L,k}(h)}{J_{O,k}(h)} = \sum_{\epsilon=\pm 1}\sum_{r=-(k-1)/2}^{(k-1)/2} \epsilon e^{hmpr^2+hr(m+\epsilon p)+\epsilon h/2},$$

where O is the unknot with zero framing, and

$$J_{O,k}(h) = \sinh(kh/2)/\sinh(h/2).$$

According to the H. Murakami and J. Murakami's result [6], the quantum dilogarithm invariant $\langle L \rangle_k$ is the following specification of the colored Jones invariant (up to a multiple of an k-th root of unity):

$$(2) \qquad \langle L \rangle_k \equiv \lim_{h \to 2\pi i/k} \frac{J_{L,k}(h)}{J_{O,k}(h)}.$$

In what follows, we shall call this as "hyperbolic specification". Our result describes the asymtotic expansion of $\langle L \rangle_k$ when $k \to \infty$.

Theorem. *The hyperbolic specification (2) of the colored Jones invariant for the (m, p) torus knot L has the following asymtotic expansion at large k:*

$$(3) \qquad \langle L \rangle_k e^{\frac{i\pi}{2k} \left(\frac{m}{p} + \frac{p}{m} \right)} = \sum_{j=1}^{mp-1} \langle L \rangle_k^{(j)} + \langle L \rangle_k^{(\infty)},$$

where

$$(4) \quad \langle L \rangle_k^{(j)} = 2(2mp/k)^{-3/2} e^{i\frac{\pi}{4}} (-1)^{(k-1)j} e^{-i\frac{\pi k j^2}{2mp}} j^2 \sin(\pi j/m) \sin(\pi j/p),$$

and

$$(5) \qquad \langle L \rangle_k^{(\infty)} = \frac{1}{4} e^{i\pi kmp/2} \sum_{n \geq 1} \frac{1}{n!} \left(\frac{i\pi}{2kmp} \right)^{n-1} \left. \frac{\partial^{2n}(x\tau_L(x))}{\partial x^{2n}} \right|_{x=0},$$

see formula (6) below for the definition of the function $\tau_L(x)$.

From eqns (3)–(5) it is easily seen that $|\langle L \rangle_k| \sim k^{3/2}$, $k \to \infty$.

Corollary. *The volume conjecture holds true for all torus knots, i.e.*

$$\lim_{k \to \infty} k^{-1} \log |\langle L \rangle_k| = 0.$$

In the next section we prove the Theorem by using an integral representation for the Gaussian sum in formula (1).

2. Proof of the Theorem

To begin with, define the following function:

$$(6) \qquad \tau_L(z) \equiv 2\sinh(mz)\sinh(pz)/\sinh(mpz).$$

It is related to the Alexander polynomial of the knot L,

$$\Delta_L(t) \equiv \frac{(t^{mp/2} - t^{-mp/2})(t^{1/2} - t^{-1/2})}{(t^{m/2} - t^{-m/2})(t^{p/2} - t^{-p/2})},$$

through the formula

$$\tau_L(z) = 2\sinh(z)/\Delta_L(e^{2z}).$$

According to the result of Milnor [4] and Turaev [9], the function $\tau_L(z)$ describes the Reidemeister torsion of the knot complement.

Lemma 1. *For any real ϕ, satisfying the condition $\mathfrak{Rhe}^{-2i\phi} > 0$, formula (1) has the following integral representation*

$$(7) \quad 2\sinh(kh/2)\frac{J_{L,k}(h)}{J_{O,k}(h)} = \sqrt{\frac{mp}{\pi h}} e^{-\frac{h}{4}\left(\frac{m}{p}+\frac{p}{m}\right)} \int_{C_\phi} dz\, e^{mp\left(kz-\frac{z^2}{h}\right)} \tau_L(z),$$

where the integration path C_ϕ is the image of the real line under the mapping

$$(8) \quad \mathbb{R} \ni x \mapsto xe^{i\phi} \in C_\phi \subset \mathbb{C},$$

with the induced orientation.

Proof. First note that for any complex $h \neq 0$ and any complex w, the following Gaussian integral formula holds:

$$(9) \quad \sqrt{\pi h}e^{hw^2} = \int_{C_\phi} dz\, e^{-z^2/h+2wz},$$

where the choice of the integration path C_ϕ, described in the formulation of the theorem, is dictated by the convergence condition of the integral, and the square root is the analytical continuation from positive values of h. Now, starting from the right hand side of eqn (1), collect the terms, containing the summation variable r, into a complete square:

$$e^{\frac{h}{4}\left(\frac{m}{p}+\frac{p}{m}\right)} \text{r.h.s.}(1) = \sum_{\epsilon=\pm 1} \epsilon \sum_{r=0}^{k-1} e^{hmp\left(r-\frac{k-1}{2}+\frac{m+\epsilon p}{2mp}\right)^2}$$

— now formula (9) can be applied to the r-dependent exponential —

$$= \sum_{\epsilon=\pm 1} \epsilon \sum_{r=0}^{k-1} \frac{1}{\sqrt{\pi hmp}} \int_{C_\phi} dz\, e^{-\frac{z^2}{hmp}+z(2r-k+1+p^{-1}+\epsilon m^{-1})}$$

— with subsequent evaluation of the summations —

$$= \frac{2}{\sqrt{\pi hmp}} \int_{C_\phi} dz\, e^{-\frac{z^2}{hmp}+z/p} \sinh(kz)\sinh(z/m)/\sinh(z)$$

— the exponential $\exp(z/p)$ in the integrand, being multiplied by an odd function of z (w.r.t. $z \leftrightarrow -z$), can by replaced by it's odd part —

$$= \frac{2}{\sqrt{\pi hmp}} \int_{C_\phi} dz\, e^{-\frac{z^2}{hmp}} \sinh(kz)\sinh(z/m)\sinh(z/p)/\sinh(z)$$

— reversing the previous argument, replace $\sinh(kz)$ by an exponential —

$$= \frac{2}{\sqrt{\pi hmp}} \int_{C_\phi} dz\, e^{-\frac{z^2}{hmp}+kz} \sinh(z/m)\sinh(z/p)/\sinh(z)$$

— and rescale the integration variable $(z \to zmp)$ —

$$= \sqrt{\frac{mp}{\pi h}} \int_{C_\phi} dz\, e^{mp\left(kz - \frac{z^2}{h}\right)} \tau_L(z)$$

with notation (6) being used.   □

Representation (7) is similar to Rozansky's formula (2.2) from [8], though the latter is only a shorthand for the power series expansion.

Lemma 2. *The hyperbolic specification (2) of the colored Jones invariant for torus knots has the integral representation*

$$(10)\quad 2\langle L\rangle_k = (mpk/2)^{3/2} e^{-\frac{i\pi}{2k}\left(\frac{m}{p} + \frac{p}{m} + \frac{k}{2}\right)} \int_{C_\phi} dz\, e^{\pi mpk(z + \frac{i}{2}z^2)} z^2 \tau_L(\pi z),$$

where integration path C_ϕ is defined in (8) with $0 < \phi < \pi/2$.

Proof. The left hand side of eqn (7) vanishes at $h = 2\pi i/k$ due to the factor $\sinh(kh/2)$. This means that the integral in the right hand side vanishes as well. So, differentiating simultaneously the sinh-function in the left hand side and the integral in the right of eqn (7) with respect to h, then putting $h = 2\pi i/k$, and rescaling the integration variable by π, we rewrite the result in the form of eqn (10).   □

Proof of the Theorem. At large k one can use the steepest descent method for evaluation the integral in (10). The only stationary point at $z = i$ is separated from the integration path by a finite number of poles of the function $\tau_L(\pi z)$ which are located at $z_j \equiv ij/mp$, $0 < j < mp$. Thus, taking into account convergence at infinity, we can shift path C_ϕ by imaginary unit and add integration along a closed contour encircling points z_j in the counterclockwise direction. The integration along the shifted path C_ϕ can be transformed by the change of the integration variable $z \to z + i$:

$$\int_{i + C_\phi} dz\, e^{\pi mpk(z + \frac{i}{2}z^2)} z^2 \tau_L(\pi z)$$

$$= e^{i\pi mpk/2} \int_{C_\phi} dz\, e^{i\pi mpkz^2/2}(z + i)^2 \tau_L(\pi z + i\pi)$$

$$= -2i e^{i\pi mpk/2} \int_{C_\phi} dz\, e^{i\pi mpkz^2/2} z \tau_L(\pi z),$$

where in the last line we have used the (quasi) periodicity property of the sinh-function, the fact that m and p are mutually prime, and disregarded the odd terms with respect to the sign change $z \leftrightarrow -z$. Now, the obtained formula straightforwardly leads to the asymptotic power series $\langle L\rangle_k^{(\infty)}$ in eqn (3) through the Taylor series expansion of the function $z\tau_L(\pi z)$ at $z = 0$, and evaluation of the Gaussian integrals. The other terms in eqn (3) come from the evaluation of the contour integral by the residue method.   □

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